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 Spin 1/2 particle in the field of the Dirac string on the
 background of de Sitter space-time

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The Dirac monopole string is specified for de Sitter cosmological model. Dirac equation for spin 1/2 particle in presence of this monopole has been examined on the background of de Sitter space-time in static coordinates. Instead of spinor monopole harmonics, the technique of Wigner D -functions is used. After separation of the variables, detailed analysis of the radial equations is performed; four types of solutions, singular, regular, in- and out- running waves, are constructed in terms of hypergeometric functions. The complete set of spinor wave solutions $\Psi_{\epsilon,j,m,\lambda}(t, r, \theta, \phi)$ has been constructed, special attention is given to treating the states of minimal values of the total angular momentum j_{\min} .

1 Introduction

De Sitter and anti de Sitter geometrical models are given steady attention in the context of developing quantum theory in a curved space-time – for instance, see in [1,2]. In particular, the problem of description of the particles with different spins on these curved backgrounds has a long history – see [3–36]. Here we will be interested mostly in treating the Dirac equation in de Sitter model.

In the present paper, the influence of the Dirac monopole string on the spin 1/2 particle in de Sitter cosmological model is investigated¹. Instead of spinor monopole harmonics, the technique of Wigner D -functions is used. After separation of the variables radial equation have been solved exactly in terms of hypergeometric functions. The complete set of spinor wave

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¹Such a problem for spinless particle in the flat Minkowski space was first considered by Dirac [37] and Tamm [38]; Harish-Chandra [39] obtained the exact solution of Dirac equation for electron interacting with magnetic-monopole field.

solutions $\Psi_{\epsilon,j,m,\lambda}(t, r, \theta, \phi)$ has been constructed. Special attention is given to treating the states of minimal values for total angular momentum quantum number j_{\min} , these states turn to be much more complicated than in the flat Minkowski space.

2 Dirac particle in de Sitter space

The Dirac equation (the notation according to [44] is used)

$$\left[i\gamma^c (e_{(c)}^\alpha \partial_\alpha + \frac{1}{2} \sigma^{ab} \gamma_{abc}) - M \right] \Psi = 0 \quad (1)$$

in static coordinates and tetrad of the Sitter space

$$\begin{aligned} dS^2 &= \Phi dt^2 - \frac{dr^2}{\Phi} - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad \Phi = 1 - r^2, \\ e_{(0)}^\alpha &= \left(\frac{1}{\sqrt{\Phi}}, 0, 0, 0 \right), \quad e_{(3)}^\alpha = (0, \sqrt{\Phi}, 0, 0), \\ e_{(1)}^\alpha &= \left(0, 0, \frac{1}{r}, 0 \right), \quad e_{(2)}^\alpha = \left(1, 0, 0, \frac{1}{r \sin \theta} \right), \\ \gamma_{030} &= \frac{\Phi'}{2\sqrt{\Phi}}, \quad \gamma_{311} = \frac{\sqrt{\Phi}}{r}, \quad \gamma_{322} = \frac{\sqrt{\Phi}}{r}, \quad \gamma_{122} = \frac{\cos \theta}{r \sin \theta}, \end{aligned} \quad (2)$$

takes the form

$$\begin{aligned} &\left[i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i\sqrt{\Phi} \left(\gamma^3 \partial_r + \frac{\gamma^1 \sigma^{31} + \gamma^2 j^{32}}{r} \right. \right. \\ &\quad \left. \left. + \frac{\Phi'}{2\Phi} \gamma^0 \sigma^{03} \right) + \frac{1}{r} \Sigma_{\theta, \phi} - M \right] \Psi(x) = 0, \end{aligned} \quad (3)$$

where

$$\Sigma_{\theta, \phi} = i \gamma^1 \partial_\theta + \gamma^2 \frac{i\partial + i\sigma^{12} \cos \theta}{\sin \theta}.$$

Eq. (3) reads

$$\left[i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i\sqrt{\Phi} \gamma^3 \left(\partial_r + \frac{1}{r} + \frac{\Phi'}{4\Phi} \right) + \frac{1}{r} \Sigma_{\theta, \phi} - M \right] \Psi(x) = 0. \quad (4)$$

With the substitution $\Psi(x) = r^{-1} \Phi^{-1/4} \psi(x)$, we get

$$\left(i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i\sqrt{\Phi} \gamma^3 \partial_r + \frac{1}{r} \Sigma_{\theta, \phi} - M \right) \psi(x) = 0. \quad (5)$$

Below the spinor basis will be used

$$\gamma^0 = \begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}, \quad \gamma^j = \begin{vmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{vmatrix}, \quad i\sigma^{12} = \begin{vmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{vmatrix}.$$

3 Separation of the variables

Let us introduce a Dirac string potential in the de Sitter space-time model. It is convenient to start with the monopole Abelian potential in the Schwinger's form for the flat Minkowski space [40]

$$A^a(x) = (A^0, A^i) = \left(0, g \frac{(\vec{r} \times \vec{n}) \cdot (\vec{r} \vec{n})}{r (r^2 - (\vec{r} \vec{n})^2)} \right). \quad (6)$$

Specifying $\vec{n} = (0, 0, 1)$ and translating the $A_\alpha(x)$ to the spherical coordinates, we get

$$A_0 = 0, \quad A_r = 0, \quad A_\theta = 0, \quad A_\phi = g \cos \theta. \quad (7)$$

This potential A_ϕ obeys Maxwell equations in de Sitter space

$$\begin{aligned} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} F^{\alpha\beta} &= 0, \quad \sqrt{-g} = r^2 \sin \theta, \\ F_{\phi\theta} &= g \sin \theta, \quad \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} r^2 \sin \theta \frac{1}{r^2} \frac{1}{r^2 \sin^2 \theta} g \sin \theta = 0. \end{aligned} \quad (8)$$

Correspondingly, the Dirac equation in presence of this field A_ϕ takes the form

$$\left(i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 \partial_r + \frac{1}{r} \Sigma_{\theta,\phi}^k - M \right) \psi(x) = 0, \quad (9)$$

where (below the notation $eg/\hbar c = k$ will be used)

$$\Sigma_{\theta,\phi}^k = i \gamma^1 \partial_\theta + \gamma^2 \frac{i \partial_\phi + (i \sigma^{12} - k) \cos \theta}{\sin \theta}. \quad (10)$$

As readily verified, the wave operator in (9) commutes with the following three ones

$$\begin{aligned} J_1^k &= l_1 + \frac{(i \sigma^{12} - k) \cos \phi}{\sin \theta}, \\ J_2^k &= l_2 + \frac{(i \sigma^{12} - k) \sin \phi}{\sin \theta}, \quad J_3^k = l_3 \end{aligned} \quad (11)$$

which in turn obey the $su(2)$ Lie algebra. Clearly, this monopole situation comes entirely under the Schrödinger [41], and Pauli [42] approach; detailed treatment of the method was given recently in [45]; similar technique when treating the problem of any spin particle in magnetic pole was used previously in [46], though with no connection with tetrad formalism.

Corresponding to diagonalization of the \vec{J}_k^2 and J_3^k , the function ψ is to be taken as ($D_\sigma \equiv D_{-m,\sigma}^j(\phi, \theta, 0)$ stands for Wigner functions [43])

$$\psi_{\epsilon jm}^k(t, r, \theta, \phi) = \frac{e^{-i\epsilon t}}{r} \begin{vmatrix} f_1 & D_{k-1/2} \\ f_2 & D_{k+1/2} \\ f_3 & D_{k-1/2} \\ f_4 & D_{k+1/2} \end{vmatrix}. \quad (12)$$

Further, with the use of recursive relations [43]

$$\begin{aligned}\partial_\theta D_{k+1/2} &= a D_{k-1/2} - b D_{k+3/2}, \quad \partial_\theta D_{k-1/2} = c D_{k-3/2} - a D_{k+1/2}, \\ \sin^{-1} \theta [-m - (k+1/2) \cos \theta] D_{k+1/2} &= -a D_{k-1/2} - b D_{k+3/2}, \\ \sin^{-1} \theta [-m - (k-1/2) \cos \theta] D_{k-1/2} &= -c D_{k-3/2} - a D_{k+1/2},\end{aligned}$$

where

$$\begin{aligned}a &= \frac{1}{2} \sqrt{(j+1/2)^2 - k^2}, \\ b &= \frac{\sqrt{(j-k-1/2)(j+k+3/2)}}{2}, \\ c &= \frac{\sqrt{(j+k-1/2)(j-k+3/2)}}{2},\end{aligned}$$

we find how the $\Sigma_{\theta,\phi}^k$ acts on $\psi_{\epsilon jm}^k$

$$\Sigma_{\theta,\phi}^k \psi_{\epsilon jm}^k = i \sqrt{(j+1/2)^2 - k^2} e^{-i\epsilon t} \begin{vmatrix} -f_4 D_{k-1/2} \\ +f_3 D_{k+1/2} \\ +f_2 D_{k-1/2} \\ -f_1 D_{k+1/2} \end{vmatrix}; \quad (13)$$

hereafter the factor $\sqrt{(j+1/2)^2 - k^2}$ will be referred to as ν . For the $f_i(r)$, the radial system derived is

$$\begin{aligned}\frac{\epsilon}{\sqrt{\Phi}} f_3 - i \sqrt{\Phi} \frac{d}{dr} f_3 - i \frac{\nu}{r} f_4 - M f_1 &= 0, \\ \frac{\epsilon}{\sqrt{\Phi}} f_4 + i \sqrt{\Phi} \frac{d}{dr} f_4 + i \frac{\nu}{r} f_3 - M f_2 &= 0, \\ \frac{\epsilon}{\sqrt{\Phi}} f_1 + i \sqrt{\Phi} \frac{d}{dr} f_1 + i \frac{\nu}{r} f_2 - M f_3 &= 0, \\ \frac{\epsilon}{\sqrt{\Phi}} f_2 - i \sqrt{\Phi} \frac{d}{dr} f_2 - i \frac{\nu}{r} f_1 - M f_4 &= 0.\end{aligned} \quad (14)$$

Else one operator can be diagonalized together with $i\partial_t, \vec{J}_k^2, J_3^k$: namely, a generalized Dirac operator

$$\hat{K}^k = -i \gamma^0 \gamma^3 \Sigma_{\theta,\phi}^k. \quad (15)$$

From the eigenvalue equation $\hat{K}^k \psi_{\epsilon jm} = \lambda \psi_{\epsilon jm}$ we can produce two possible values for this λ and the corresponding restrictions on $f_i(r)$

$$\lambda = -\delta \sqrt{(j+1/2)^2 - k^2}, \quad f_4 = \delta f_1, \quad f_3 = \delta f_2. \quad (16)$$

Correspondingly, the system (14) reduces to

$$\begin{aligned}\left(\sqrt{\Phi} \frac{d}{dr} + \frac{\nu}{r} \right) f + \left(\frac{\epsilon}{\sqrt{\Phi}} + \delta M \right) g &= 0, \\ \left(\sqrt{\Phi} \frac{d}{dr} - \frac{\nu}{r} \right) g - \left(\frac{\epsilon}{\sqrt{\Phi}} - \delta M \right) f &= 0,\end{aligned} \quad (17)$$

to exclude imaginary i we have translated equations to new functions

$$f = \frac{f_1 + f_2}{\sqrt{2}}, \quad g = \frac{f_1 - f_2}{i\sqrt{2}}.$$

Note the quantization rule for $k = eg/\hbar c$ and j

$$\begin{aligned} \frac{eg}{\hbar c} &= \pm 1/2, \pm 1, \pm 3/2, \dots; \\ j &= |k| - 1/2, |k| + 1/2, |k| + 3/2, \dots \end{aligned} \quad (18)$$

The case of minimal value $j_{\min} = |k| - 1/2$ must be separated and treated in a special way. For example, let $k = +1/2$, then to the minimal value $j = 0$ there corresponds a wave function in terms of only (t, r) -dependent quantities

$$\psi_{k=+1/2}^{(j=0)}(x) = e^{-iet} \begin{vmatrix} f_1(r) \\ 0 \\ f_3(r) \\ 0 \end{vmatrix}; \quad (19)$$

at $k = -1/2$, we have

$$\psi_{k=-1/2}^{(j=0)}(x) = e^{-iet} \begin{vmatrix} 0 \\ f_2(r) \\ 0 \\ f_4(r) \end{vmatrix}. \quad (20)$$

Thus, if $k = \pm 1/2$, then to the minimal values j_{\min} there correspond the substitutions which do not depend at all on the angular variables (θ, ϕ) . At this point there exists some formal analogy between the electron-monopole states and S -states (with $l = 0$) for a boson field of spin zero: $\Phi_{l=0} = \Phi(r, t)$. However, it would be unwise to attach too much significance to this formal coincidence because that (θ, ϕ) -independence of $(e - g)$ -states is not the fact invariant under tetrad gauge transformations. In contrast, the relation below (let $k = +1/2$)

$$\Sigma_{\theta, \phi}^{+1/2} \psi_{k=+1/2}^{(j=0)}(x) = \gamma^2 \cot \theta (i\sigma^{12} - 1/2) \psi_{k=+1/2}^{(j=0)} \equiv 0 \quad (21)$$

is invariant under any gauge transformations. The identity (21) holds because all the zeros in the $\psi_{k=+1/2}^{(j=0)}$ are adjusted to the non-zeros in $(i\sigma^{12} - 1/2)$ and conversely; the non-vanishing constituents in $\psi_{k=+1/2}^{(j=0)}$ are canceled out by zeros in $(i\sigma^{12} - 1/2)$. Correspondingly, the matter equation (9) assumes the more simple form

$$\left(i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \gamma^3 \sqrt{\Phi} \partial_r - M \right) \psi^{(j=0)} = 0. \quad (22)$$

It is readily verified that both (19) and (20) representations are extended to $(e - g)$ -states

with $j = j_{\min}$ at all other $k = \pm 1, \pm 3/2, \dots$. Indeed,

$$k = +1, +3/2, +2, \dots, \quad \psi_{j_{\min}}^{k>0}(x) = e^{-i\epsilon t} \begin{vmatrix} f_1(r) & D_{k-1/2} \\ 0 & f_3(r) & D_{k-1/2} \\ 0 & 0 \end{vmatrix}; \quad (23)$$

$$k = -1, -3/2, -2, \dots, \quad \psi_{j_{\min}}^{k<0}(x) = e^{-i\epsilon t} \begin{vmatrix} 0 & f_2(r) & D_{k+1/2} \\ 0 & 0 & f_4(r) & D_{k+1/2} \end{vmatrix}, \quad (24)$$

and the relation $\Sigma_{\theta,\phi} \Psi_{j_{\min}} = 0$ still holds. For instance, let us consider in more detail the case of positive k . Using the recursive relations

$$\begin{aligned} \partial_\theta D_{k-1/2} &= \frac{1}{2} \sqrt{2k-1} D_{k-3/2}, \\ \sin^{-1} \theta [-m - (k-1/2) \cos \theta] D_{k-1/2} &= -\frac{1}{2} \sqrt{2k-1} D_{k-3/2}, \end{aligned}$$

we get

$$\begin{aligned} i\gamma^1 \partial_\theta \begin{vmatrix} f_1 D_{k-1/2} \\ 0 \\ f_3 D_{k-1/2} \\ 0 \end{vmatrix} &= \frac{i}{2} \sqrt{2k-1} \begin{vmatrix} 0 \\ -f_3 D_{k-3/2} \\ 0 \\ +f_1 D_{k-3/2} \end{vmatrix}, \\ \gamma^2 \frac{i\partial_\phi + (i\sigma^{12} - k) \cos \theta}{\sin \theta} \begin{vmatrix} f_1 D_{k-1/2} \\ 0 \\ f_3 D_{k-1/2} \\ 0 \end{vmatrix} &= \frac{i}{2} \sqrt{2k-1} \begin{vmatrix} 0 \\ +f_3 D_{k-3/2} \\ 0 \\ -f_1 D_{k-3/2} \end{vmatrix}; \end{aligned}$$

in a sequence, the identity $\Sigma_{\theta,\phi} \psi_{j_{\min}} \equiv 0$ takes place. The case of negative k can be considered in the same way. As regards the operator \hat{K}^k , for the j_{\min} states we get $\hat{K}^k \psi_{j_{\min}} = 0$.

Thus, at every k , the j_{\min} -equation has the same form

$$\left(i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i\gamma^3 \sqrt{\Phi} \partial_r + \frac{1}{r} \right) \psi_{j_{\min}} = 0; \quad (25)$$

which leads to the same radial system

$$k = +1/2, +1, \dots$$

$$\begin{aligned} \frac{\epsilon}{\sqrt{\Phi}} f_3 - i \sqrt{\Phi} \frac{d}{dr} f_3 - M f_1 &= 0, \\ \frac{\epsilon}{\sqrt{\Phi}} f_1 + i \sqrt{\Phi} \frac{d}{dr} f_1 - M f_3 &= 0; \end{aligned} \quad (26)$$

$$k = -1/2, -1, \dots$$

$$\begin{aligned} \frac{\epsilon}{\sqrt{\Phi}} f_4 + i \sqrt{\Phi} \frac{d}{dr} f_4 - M f_2 &= 0, \\ \frac{\epsilon}{\sqrt{\Phi}} f_2 - i \sqrt{\Phi} \frac{d}{dr} f_2 - M f_4 &= 0. \end{aligned} \quad (27)$$

In the limit of the flat space-time, these equations are equivalent respectively to

$$k = +1/2, +1, \dots$$

$$\left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 \right) f_1 = 0, \quad f_3 = \frac{1}{m} \left(\epsilon + i \frac{d}{dr} \right) f_1; \quad (28)$$

$$k = -1/2, -1, \dots$$

$$\left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 \right) f_4 = 0, \quad f_2 = \frac{1}{m} \left(\epsilon + i \frac{d}{dr} \right) f_4. \quad (29)$$

These equations end up with the functions $f = \exp(\pm \sqrt{m^2 - \epsilon^2} r)$. In particular, at $\epsilon < m$, there arise solutions of the form

$$\exp(-\sqrt{m^2 - \epsilon^2} r), \quad (30)$$

which seem to be appropriate to describe bound state s in the electron-monopole system. It should be emphasized that today the j_{\min} bound state problem remains still yet a question to understand. In particular, the important question is of finding a physical and mathematical criterion on selecting values for ϵ : whether $\epsilon < m$, or $\epsilon = m$, or $\epsilon > m$; and which value of ϵ is to be chosen after specifying an interval above. The case $\epsilon = m$ is the most special one – it gives $f_1 = f_2 = 1$ and $f_4 = f_2 = 1$.

4 Solution of the radial equations

Let us turn back to eqs. (17); for definiteness we will consider the case $\delta = +1$ (the case $\delta = -1$ follows from the former through the formal change $M \Rightarrow -M$)

$$\begin{aligned} \left(\sqrt{\Phi} \frac{d}{dr} + \frac{\nu}{r} \right) f + \left(\frac{\epsilon}{\sqrt{\Phi}} + M \right) g &= 0, \\ \left(\sqrt{\Phi} \frac{d}{dr} - \frac{\nu}{r} \right) g - \left(\frac{\epsilon}{\sqrt{\Phi}} - M \right) f &= 0. \end{aligned} \quad (31)$$

Here we note additional singularities at the points

$$\epsilon + \sqrt{\Phi} M = 0 \quad \text{or} \quad \epsilon - \sqrt{\Phi} M = 0.$$

For instance, the equation for $f(r)$ has the form

$$\begin{aligned} \frac{d^2}{dr^2} f - \left(\frac{2r}{1-r^2} - \frac{Mr}{\sqrt{1-r^2}(\epsilon + M\sqrt{1-r^2})} \right) \frac{d}{dr} f + \left(\frac{\epsilon^2}{(1-r^2)^2} - \frac{M^2}{1-r^2} \right. \\ \left. - \frac{\nu(\nu+1)}{r^2(1-r^2)} - \frac{\nu}{(1-r^2)\sqrt{1-r^2}} + \frac{M\nu}{\sqrt{1-r^2}(\epsilon + M\sqrt{1-r^2})} \right) f = 0. \end{aligned}$$

However, there exists possibility to move these singularities away through a special transformation of the functions $f(r), g(r)$ (see in [24]). To this end, first let us introduce a new variable

$r = \sin \rho$, eqs. (31) look simpler

$$\begin{aligned} \left(\frac{d}{d\rho} + \frac{\nu}{\sin \rho} \right) f + \left(\frac{\epsilon}{\cos \rho} + M \right) g &= 0, \\ \left(\frac{d}{d\rho} - \frac{\nu}{\sin \rho} \right) g - \left(\frac{\epsilon}{\cos \rho} - M \right) f &= 0. \end{aligned} \quad (32)$$

Summing and subtracting two last equations, we get

$$\begin{aligned} \frac{d}{d\rho}(f+g) + \frac{\nu}{\sin \rho}(f-g) - \frac{\epsilon}{\cos \rho}(f-g) + M(f+g) &= 0, \\ \frac{d}{d\rho}(f-g) + \frac{\nu}{\sin \rho}(f+g) + \frac{\epsilon}{\cos \rho}(f+g) - M(f-g) &= 0. \end{aligned} \quad (33)$$

Introducing two new functions

$$f+g = e^{-i\rho/2}(F+G), \quad f-g = e^{+i\rho/2}(F-G), \quad (34)$$

or in matrix form

$$\begin{vmatrix} f \\ g \end{vmatrix} = \begin{vmatrix} \cos \frac{\rho}{2} & -i \sin \frac{\rho}{2} \\ -i \sin \frac{\rho}{2} & \cos \frac{\rho}{2} \end{vmatrix} \begin{vmatrix} F \\ G \end{vmatrix} \quad (35)$$

one transforms (33) into

$$\begin{aligned} \frac{d}{d\rho} e^{-i\rho/2}(F+G) + \frac{\nu}{\sin \rho} e^{+i\rho/2}(F-G) \\ - \frac{\epsilon}{\cos \rho} e^{+i\rho/2}(F-G) + M e^{-i\rho/2}(F+G) &= 0, \\ \frac{d}{d\rho} e^{+i\rho/2}(F-G) + \frac{\nu}{\sin \rho} e^{-i\rho/2}(F+G) \\ + \frac{\epsilon}{\cos \rho} e^{-i\rho/2}(F+G) - M e^{+i\rho/2}(F-G) &= 0, \end{aligned}$$

or

$$\begin{aligned} \frac{d}{d\rho}(F+G) - \frac{i}{2}(F+G) + \frac{\nu}{\sin \rho}(\cos \rho + i \sin \rho)(F-G) \\ - \frac{\epsilon}{\cos \rho}(\cos \rho + i \sin \rho)(F-G) + M(F+G) &= 0, \\ \frac{d}{d\rho}(F-G) + \frac{i}{2}(F-G) + \frac{\nu}{\sin \rho}(\cos \rho - i \sin \rho)(F+G) \\ + \frac{\epsilon}{\cos \rho}(\cos \rho - i \sin \rho)(F+G) - M(F-G) &= 0. \end{aligned}$$

Finally, summing and subtracting two last equations

$$\begin{aligned} \left(\frac{d}{d\rho} + \nu \frac{\cos \rho}{\sin \rho} - i\epsilon \frac{\sin \rho}{\cos \rho} \right) F + \left(\epsilon + M - i\nu - \frac{i}{2} \right) G &= 0, \\ \left(\frac{d}{d\rho} - \nu \frac{\cos \rho}{\sin \rho} + i\epsilon \frac{\sin \rho}{\cos \rho} \right) G + \left(-\epsilon + M + i\nu - \frac{i}{2} \right) F &= 0. \end{aligned} \quad (36)$$

The equations produced have no singular points, except $\rho = 0, \pi/2$. It should be noted that for the second class of states with $\delta = -1$, the relevant system differ from (39) in the sign of M : $M \Rightarrow -M$.

Let us translate the system (39) to the variable

$$z = r^2 = \sin^2 \rho, \quad z \in [0, +1) ; \quad (37)$$

$$\begin{vmatrix} f \\ g \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{1+\sqrt{1-z}}{2}} & -i\sqrt{\frac{1-\sqrt{1-z}}{2}} \\ -i\sqrt{\frac{1-\sqrt{1-z}}{2}} & \sqrt{\frac{1+\sqrt{1-z}}{2}} \end{vmatrix} \begin{vmatrix} F \\ G \end{vmatrix}, \quad (38)$$

so we get

$$\begin{aligned} (2\sqrt{z(1-z)} \frac{d}{dz} + \nu \frac{\sqrt{1-z}}{\sqrt{z}} - i\epsilon \frac{\sqrt{z}}{\sqrt{1-z}})F \\ + (+\epsilon + M - i\nu - \frac{i}{2})G = 0, \\ (2\sqrt{z(1-z)} \frac{d}{dz} - \nu \frac{\sqrt{1-z}}{\sqrt{z}} + i\epsilon \frac{\sqrt{z}}{\sqrt{1-z}})G \\ + (-\epsilon + M + i\nu - \frac{i}{2})F = 0. \end{aligned} \quad (39)$$

From (39) it follow 2-nd order differential equations for F and G

$$\begin{aligned} z(1-z) \frac{d^2 F}{dz^2} + (\frac{1}{2} - z) \frac{dF}{dz} \\ + \left[-\frac{1}{4} \left(M - \frac{i}{2} \right)^2 + \frac{\epsilon(\epsilon - i)}{4(1-z)} - \frac{\nu(\nu + 1)}{4z} \right] F = 0, \\ z(1-z) \frac{d^2 G}{dz^2} + (\frac{1}{2} - z) \frac{dG}{dz} \\ + \left[-\frac{1}{4} \left(M - \frac{i}{2} \right)^2 + \frac{\epsilon(\epsilon + i)}{4(1-z)} - \frac{\nu(\nu - 1)}{4z} \right] G = 0. \end{aligned} \quad (40)$$

It should be noted symmetry between two equations according to formal changes

$$\nu \longrightarrow -\nu, \quad \epsilon \longrightarrow -\epsilon. \quad (41)$$

Let us introduce substitutions

$$F = z^A (1-z)^B \bar{F}(z), \quad G = z^K (1-z)^L \bar{G}(z),$$

eqs. (40) give

$$\begin{aligned} z(1-z) \frac{d^2 \bar{F}}{dz^2} + \left[2A + \frac{1}{2} - (2A + 2B + 1)z \right] \frac{d\bar{F}}{dz} \\ + \left[-\frac{1}{4} \left(M - \frac{i}{2} \right)^2 - (A+B)^2 + \frac{\epsilon(\epsilon - i) + 2B(2B - 1)}{4(1-z)} \right. \\ \left. - \frac{\nu(\nu + 1) - 2A(2A - 1)}{4z} \right] \bar{F} = 0, \end{aligned} \quad (42)$$

$$\begin{aligned}
& z(1-z) \frac{d^2 \bar{G}}{dz^2} + \left[2K + \frac{1}{2} - (2K + 2L + 1)z \right] \frac{d\bar{G}}{dz} \\
& + \left[-\frac{1}{4} \left(M - \frac{i}{2} \right)^2 - (K + L)^2 + \frac{\epsilon(\epsilon + i) + 2L(2L - 1)}{4(1 - z)} \right. \\
& \quad \left. - \frac{\nu(\nu - 1) - 2K(2K - 1)}{4z} \right] \bar{G} = 0.
\end{aligned} \tag{43}$$

First let us consider eq. (42); at A, B taken accordingly

$$A = \frac{1 + \nu}{2}, -\frac{\nu}{2}, \quad B = -\frac{i\epsilon}{2}, \frac{1 + i\epsilon}{2} \tag{44}$$

it becomes simpler

$$\begin{aligned}
& z(1-z) \frac{d^2 \bar{F}}{dz^2} + \left[2A + \frac{1}{2} - (2A + 2B + 1)z \right] \frac{d\bar{F}}{dz} \\
& + \left[-\frac{1}{4} \left(M - \frac{i}{2} \right)^2 - (A + B)^2 \right] \bar{F} = 0,
\end{aligned} \tag{45}$$

which is of hypergeometric type with parameters

$$a = A + B + \frac{iM + 1/2}{2}, \quad b = A + B - \frac{iM + 1/2}{2}, \quad c = 2A + 1/2.$$

To have solutions regular in the origin $z = 0$, we should take positive A . Thus there arise two sorts of solutions depending on a chosen B (in each case two linearly independent solutions, regular and singular in the origin, are written down):

the first

$$\begin{aligned}
A + B &= \frac{1 + \nu - i\epsilon}{2} \quad c = \nu + 3/2, \\
\bar{F}_{reg}^{(1)}(z) &= F(a, b, c; z), \\
\bar{F}_{sing}^{(1)}(z) &= z^{1-c} F(a + 1 - c, b + 1 - c, 2 - c; z), \\
a &= \frac{1 + \nu - i\epsilon}{2} + \frac{iM + 1/2}{2}, \\
b &= \frac{1 + \nu - i\epsilon}{2} - \frac{iM + 1/2}{2};
\end{aligned} \tag{46}$$

the second

$$\begin{aligned}
A + B &= \frac{2 + \nu + i\epsilon}{2} \quad \gamma = \nu + 3/2, \\
\bar{F}_{reg}^{(2)}(z) &= F(\alpha, \beta, \gamma; z), \\
\bar{F}_{sing}^{(2)}(z) &= z^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma; z), \\
\alpha &= \frac{2 + \nu + i\epsilon}{2} + \frac{iM + 1/2}{2}, \\
\beta &= \frac{2 + \nu + i\epsilon}{2} - \frac{iM + 1/2}{2}.
\end{aligned} \tag{47}$$

Not let us turn back to eq. (43); at K and L chosen according to

$$K = \frac{1-\nu}{2}, \frac{\nu}{2}, \quad L = \frac{i\epsilon}{2}, \frac{1-i\epsilon}{2} \quad (48)$$

it will be simpler

$$\begin{aligned} z(1-z) \frac{d^2 \bar{G}}{dz^2} + \left[2K + \frac{1}{2} - (2K + 2L + 1)z \right] \frac{d\bar{G}}{dz} \\ + \left[-\frac{1}{4} \left(M - \frac{i}{2} \right)^2 - (K + L)^2 \right] \bar{G} = 0, \end{aligned} \quad (49)$$

which is of hypergeometric type

$$a' = K + L + \frac{iM + 1/2}{2}, \quad b' = K + L - \frac{iM + 1/2}{2}, \quad c' = 2K + \frac{1}{2}.$$

To have solutions regular in the origin $z = 0$, we take positive K . Thus there arise two sorts of solutions depending on a chosen B (in each case two linearly independent solutions, regular and singular in the origin, are written down)

the first

$$\begin{aligned} K + L &= \frac{\nu + i\epsilon}{2} \quad c' = \nu + 1/2, \\ \bar{G}_{reg}^{(1)}(z) &= F(a', b', c'; z), \\ \bar{G}_{sing}^{(1)}(z) &= z^{1-c'} F(a' + 1 - c', b' + 1 - c', 2 - c'; z), \\ a' &= \frac{\nu + i\epsilon}{2} + \frac{iM + 1/2}{2}, \\ b' &= \frac{\nu + i\epsilon}{2} - \frac{iM + 1/2}{2}; \end{aligned} \quad (50)$$

the second

$$\begin{aligned} K + L &= \frac{\nu + 1 - i\epsilon}{2} \quad \gamma' = \nu + 1/2, \\ \bar{G}_{reg}^{(2)}(z) &= F(\alpha', \beta', \gamma'; z), \\ \bar{G}_{sing}^{(2)}(z) &= z^{1-\gamma'} F(\alpha' + 1 - \gamma', \beta' + 1 - \gamma', 2 - \gamma'; z), \\ \alpha' &= \frac{\nu + 1 - i\epsilon}{2} + \frac{iM + 1/2}{2} = a, \\ \beta' &= \frac{\nu + 1 - i\epsilon}{2} - \frac{iM + 1/2}{2} = b. \end{aligned} \quad (51)$$

Thus, we have constructed the following four regular solutions

$$F_{reg}^{(1)}, \quad F_{reg}^{(2)}, \quad G_{reg}^{(1)}, \quad G_{reg}^{(2)}.$$

However, due to the known identity for hypergeometric functions

$$F(A, B, C; z) = (1-z)^{C-A-B} F(C-A, C-B, C; z) \quad (52)$$

we readily conclude that there exist only two different ones:

$$\begin{aligned} F_{reg}^{(1)} &= z^{(\nu+1)/2} (1-z)^{-i\epsilon/2} F(a, b, c, z) \\ &= z^{(\nu+1)/2} (1-z)^{(1+i\epsilon)/2} F(\alpha, \beta, \gamma, z) = F_{reg}^{(2)}, \end{aligned} \quad (53)$$

$$\begin{aligned} G_{reg}^{(1)} &= z^{\nu/2} (1-z)^{+i\epsilon/2} F(a', b', c', z) \\ &= z^{\nu/2} (1-z)^{(1-i\epsilon)/2} F(\alpha', \beta', \gamma', z) = G_{reg}^{(2)}. \end{aligned} \quad (54)$$

The same is true for singular solutions

$$F_{sing}^{(1)} = F_{sing}^{(2)}, \quad G_{sing}^{(1)} = G_{sing}^{(2)}. \quad (55)$$

Taking into account relations

$$\begin{aligned} \alpha = a' + 1, \quad \beta = b' + 1, \quad \gamma = c' + 1, \\ \bar{G}_{reg}^{(1)}(z) \implies \bar{F}_{reg}^{(2)}(z); \end{aligned} \quad (56)$$

we can expect an identity

$$\begin{aligned} &\left(2\sqrt{z(1-z)} \frac{d}{dz} - \nu \frac{\sqrt{1-z}}{\sqrt{z}} + i\epsilon \frac{\sqrt{z}}{\sqrt{1-z}} \right) \\ &\quad \times G_0^{reg} z^{\nu/2} (1-z)^{+i\epsilon/2} F(a', b', c', z) \\ &+ (-\epsilon + M + i\nu - \frac{i}{2}) F_0^{reg} z^{(1+\nu)/2} (1-z)^{(1+i\epsilon)/2} F(\alpha, \beta, \gamma, z) = 0. \end{aligned}$$

From whence it follows

$$2G_0^{reg} \frac{d}{dz} F(a', b', c', z) + (-\epsilon + M + i\nu - \frac{i}{2}) F_0^{reg} F(\alpha, \beta, \gamma, z) = 0, \quad (57)$$

and further

$$2G_0^{reg} \frac{a'b'}{c'} + (-\epsilon + M + i\nu - \frac{i}{2}) F_0^{reg} = 0. \quad (58)$$

In the same manner, noting that

$$\begin{aligned} (\alpha' + 1 - \gamma') &= (a + 1 - c) + 1, \\ (\beta' + 1 - \gamma') &= (b + 1 - c) + 1, \\ (2 - \gamma') &= (2 - c) + 1, \\ \bar{F}_{sing}^{(1)}(z) &\implies \bar{G}_{sing}^{(2)}(z), \end{aligned} \quad (59)$$

we can assume that

$$\begin{aligned} &\left(2\sqrt{z(1-z)} \frac{d}{dz} + \nu \frac{\sqrt{1-z}}{\sqrt{z}} - i\epsilon \frac{\sqrt{z}}{\sqrt{1-z}} \right) \\ &\quad \times F_0^{sing} z^{-\nu/2} (1-z)^{-i\epsilon/2} F(a+1-c, b+1-c, 2-c, z) \\ &\quad + (\epsilon + M - i\nu - \frac{i}{2}) z^{(1-\nu)/2} (1-z)^{(1-i\epsilon)/2} \\ &\quad \times G_0^{sing} F(\alpha' + 1 - \gamma', \beta' + 1 - \gamma', 2 - \gamma'; z) = 0. \end{aligned} \quad (60)$$

From this it follows

$$2F_0^{sing} \frac{d}{dz} F(a+1-c, b+1-c, 2-c, z) + (\epsilon + M - i\nu - \frac{i}{2}) G_0^{sing} F(\alpha' + 1 - \gamma', \beta' + 1 - \gamma', 2 - \gamma'; z) = 0 , \quad (61)$$

which leads us to

$$2F_0^{sing} \frac{(a+1-c)(b+1-c)}{2-c} + (\epsilon + M - i\nu - \frac{i}{2}) G_0^{sing} = 0$$

so that

$$F_0^{sing}(-i\epsilon - \nu + iM + 1/2) + i(1 - 2\nu)G_0^{sing} = 0 . \quad (62)$$

Thus, we have constructed regular and singular solutions of the system:

$$\begin{aligned} F_{reg}^{(1)} = F_{reg}^{(2)} = F_{reg} \quad - - \quad G_{reg} = G_{reg}^{(1)} = G_{reg}^{(2)} ; \\ F_{sing}^{(1)} = F_{sing}^{(2)} = F_{sing} \quad - - \quad G_{sing} = G_{sing}^{(1)} = G_{sing}^{(2)} . \end{aligned} \quad (63)$$

5 Radial equations in the case j_{\min}

Let us turn back to the case of the minimal value of j :

$$k = +1/2, +1, \dots$$

$$\begin{aligned} \frac{\epsilon}{\sqrt{\Phi}} f_3 - i \sqrt{\Phi} \frac{d}{dr} f_3 - M f_1 &= 0 , \\ \frac{\epsilon}{\sqrt{\Phi}} f_1 + i \sqrt{\Phi} \frac{d}{dr} f_1 - M f_3 &= 0 ; \end{aligned} \quad (64)$$

from whence for new functions

$$h = \frac{f_1 + f_3}{\sqrt{2}} , \quad g = \frac{f_1 - f_3}{i\sqrt{2}} \quad (65)$$

we derive

$$k = +1/2, +1, \dots$$

$$\sqrt{\Phi} \frac{d}{dr} h + \left(\frac{\epsilon}{\sqrt{\Phi}} + M \right) g = 0 , \quad \sqrt{\Phi} \frac{d}{dr} g - \left(\frac{\epsilon}{\sqrt{\Phi}} - M \right) h = 0 . \quad (66)$$

In the same manner for another case we have

$$k = -1/2, -1, \dots$$

$$\begin{aligned} \frac{\epsilon}{\sqrt{\Phi}} f_4 + i \sqrt{\Phi} \frac{d}{dr} f_4 - M f_2 &= 0 , \\ \frac{\epsilon}{\sqrt{\Phi}} f_2 - i \sqrt{\Phi} \frac{d}{dr} f_2 - M f_4 &= 0 ; \end{aligned} \quad (67)$$

for new functions (note difference between (65) and (68))

$$g = \frac{f_2 + f_4}{\sqrt{2}}, \quad h = \frac{f_2 - f_4}{i\sqrt{2}} \quad (68)$$

we obtain

$$\sqrt{\Phi} \frac{d}{dr} h + \left(\frac{\epsilon}{\sqrt{\Phi}} - M \right) g = 0, \quad \sqrt{\Phi} \frac{d}{dr} g - \left(\frac{\epsilon}{\sqrt{\Phi}} + M \right) h = 0. \quad (69)$$

Let us perform special transformation over the functions

$$g + h = e^{-i\rho/2}(F + G), \quad g - h = e^{+i\rho/2}(F - G). \quad (70)$$

After simple calculation we arrive at

when $k = +1/2, +3/2, \dots$

$$\begin{aligned} \left(\frac{d}{d\rho} - i\epsilon \frac{\sin \rho}{\cos \rho} \right) F + \left(\epsilon + M - \frac{i}{2} \right) G &= 0, \\ \left(\frac{d}{d\rho} + i\epsilon \frac{\sin \rho}{\cos \rho} \right) G + \left(-\epsilon + M - \frac{i}{2} \right) F &= 0; \end{aligned} \quad (71)$$

when $k = -1/2, -3/2, \dots$

$$\begin{aligned} \left(\frac{d}{d\rho} - i\epsilon \frac{\sin \rho}{\cos \rho} \right) G + \left(\epsilon - M - \frac{i}{2} \right) H &= 0, \\ \left(\frac{d}{d\rho} + i\epsilon \frac{\sin \rho}{\cos \rho} \right) H + \left(-\epsilon - M - \frac{i}{2} \right) G &= 0. \end{aligned} \quad (72)$$

The difference between (71) and (72) consists in the only change $M \longleftrightarrow -M$. The system (71) can be compared with the similar one (39)

$$\begin{aligned} \left(\frac{d}{d\rho} + \nu \frac{\cos \rho}{\sin \rho} - i\epsilon \frac{\sin \rho}{\cos \rho} \right) F + \left(\epsilon - i\nu + M - \frac{i}{2} \right) G &= 0, \\ \left(\frac{d}{d\rho} - \nu \frac{\cos \rho}{\sin \rho} + i\epsilon \frac{\sin \rho}{\cos \rho} \right) G + \left(-\epsilon + i\nu + M - \frac{i}{2} \right) F &= 0. \end{aligned} \quad (73)$$

We immediately conclude that the approach used to treat (73) can be applied here as well. In particular, the system (71) being translated to the variable z

$$\sin \rho = \sqrt{z}, \quad \cos \rho = \sqrt{1-z}, \quad \frac{d}{d\rho} = 2\sqrt{z(1-z)} \frac{d}{dz},$$

will take the form

$$\begin{aligned} \sqrt{z(1-z)} \left(\frac{d}{dz} - \frac{i\epsilon/2}{1-z} \right) F + \frac{M + \epsilon - i/2}{2} G &= 0, \\ \sqrt{z(1-z)} \left(\frac{d}{dz} + \frac{i\epsilon/2}{1-z} \right) G + \frac{M - \epsilon - i/2}{2} F &= 0. \end{aligned} \quad (74)$$

Because to states with minimal j are drawn usually great attention, let us specify these states in more detail. From (74) it follow 2-nd order differential equations for F and G respectively

$$\begin{aligned} z(1-z)\frac{d^2F}{dz^2} + \left(\frac{1}{2}-z\right)\frac{dF}{dz} + \left[-\frac{1}{4}\left(M-\frac{i}{2}\right)^2 + \frac{\epsilon(\epsilon-i)}{4(1-z)}\right]F &= 0, \\ z(1-z)\frac{d^2G}{dz^2} + \left(\frac{1}{2}-z\right)\frac{dG}{dz} + \left[-\frac{1}{4}\left(M-\frac{i}{2}\right)^2 + \frac{\epsilon(\epsilon+i)}{4(1-z)}\right]G &= 0. \end{aligned} \quad (75)$$

It should be noted symmetry between two equations according to formal changes $\epsilon \longrightarrow -\epsilon$. Let us introduce substitutions

$$F = z^A(1-z)^B\bar{F}(z), \quad G = z^K(1-z)^L\bar{G}(z),$$

eqs. (75) give

$$\begin{aligned} z(1-z)\frac{d^2\bar{F}}{dz^2} + \left[2A + \frac{1}{2} - (2A+2B+1)z\right]\frac{d\bar{F}}{dz} \\ + \left[-\frac{1}{4}\left(M-\frac{i}{2}\right)^2 - (A+B)^2 + \frac{\epsilon(\epsilon-i) + 2B(2B-1)}{4(1-z)}\right. \\ \left. + \frac{2A(2A-1)}{4z}\right]\bar{F} &= 0, \end{aligned} \quad (76)$$

$$\begin{aligned} z(1-z)\frac{d^2\bar{G}}{dz^2} + \left[2K + \frac{1}{2} - (2K+2L+1)z\right]\frac{d\bar{G}}{dz} \\ + \left[-\frac{1}{4}\left(M-\frac{i}{2}\right)^2 - (K+L)^2 + \frac{\epsilon(\epsilon+i) + 2L(2L-1)}{4(1-z)}\right. \\ \left. + \frac{2K(2K-1)}{4z}\right]\bar{G} &= 0. \end{aligned} \quad (77)$$

First let us consider eq. (76); at A and B taken accordingly

$$A = \frac{1}{2}, \quad 0, \quad B = -\frac{i\epsilon}{2}, \quad \frac{1+i\epsilon}{2} \quad (78)$$

it becomes simpler

$$\begin{aligned} z(1-z)\frac{d^2\bar{F}}{dz^2} + \left[2A + \frac{1}{2} - (2A+2B+1)z\right]\frac{d\bar{F}}{dz} \\ + \left[-\frac{1}{4}\left(M-\frac{i}{2}\right)^2 - (A+B)^2\right]\bar{F} &= 0, \end{aligned} \quad (79)$$

which is of hypergeometric type with parameters

$$a = A + B + \frac{iM+1/2}{2}, \quad b = A + B - \frac{iM+1/2}{2}, \quad c = 2A + 1/2.$$

To have solutions non-vanishing in the origin $z = 0$, we take zero $A = 0$. Thus there arise two sorts of solutions depending on a chosen B (in each case two linearly independent solutions, regular and singular in the origin, are written down)

the first

$$\begin{aligned}
A + B &= \frac{-i\epsilon}{2} \quad c = +1/2, \\
\bar{F}_{non-zero}^{(1)}(z) &= F(a, b, c; z), \\
\bar{F}_{zero}^{(1)}(z) &= z^{1-c} F(a + 1 - c, b + 1 - c, 2 - c; z), \\
a &= \frac{-i\epsilon}{2} + \frac{iM + 1/2}{2}, \quad b = \frac{-i\epsilon}{2} - \frac{iM + 1/2}{2};
\end{aligned} \tag{80}$$

the second

$$\begin{aligned}
A + B &= \frac{1 + i\epsilon}{2} \quad \gamma = +1/2, \\
\bar{F}_{non-zero}^{(2)}(z) &= F(\alpha, \beta, \gamma; z), \\
\bar{F}_{zero}^{(2)}(z) &= z^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma; z), \\
\alpha &= \frac{1 + i\epsilon}{2} + \frac{iM + 1/2}{2}, \quad \beta = \frac{1 + i\epsilon}{2} - \frac{iM + 1/2}{2};
\end{aligned} \tag{81}$$

Now let us turn back to eq. (77); at K and L chosen according to

$$K = \frac{1}{2}, 0, \quad L = \frac{i\epsilon}{2}, \frac{1 - i\epsilon}{2} \tag{82}$$

it will be simpler

$$\begin{aligned}
z(1-z) \frac{d^2 \bar{G}}{dz^2} + \left[2K + \frac{1}{2} - (2K + 2L + 1)z \right] \frac{d\bar{G}}{dz} \\
+ \left[-\frac{1}{4} \left(M - \frac{i}{2} \right)^2 - (K + L)^2 \right] \bar{G} = 0,
\end{aligned} \tag{83}$$

which is of hypergeometric type

$$a' = K + L + \frac{iM + 1/2}{2}, \quad b' = K + L - \frac{iM + 1/2}{2}, \quad c' = 2K + \frac{1}{2}.$$

We start with solutions non-vanishing in the origin $z = 0$, we take zero $K = 0$. Thus there arise two sorts of solutions depending on a chosen B (in each case two linearly independent solutions, regular and singular in the origin, are written down)

the first

$$\begin{aligned}
K + L &= \frac{+i\epsilon}{2} \quad c' = +1/2, \\
\bar{G}_{non-zero}^{(1)}(z) &= F(a', b', c'; z), \\
\bar{G}_{zero}^{(1)}(z) &= z^{1-c'} F(a' + 1 - c', b' + 1 - c', 2 - c'; z), \\
a' &= \frac{+i\epsilon}{2} + \frac{iM + 1/2}{2}, \quad b' = \frac{+i\epsilon}{2} - \frac{iM + 1/2}{2};
\end{aligned} \tag{84}$$

the second

$$\begin{aligned}
K + L &= \frac{1 - i\epsilon}{2} \quad \gamma' = +1/2, \\
\bar{G}_{non-zero}^{(2)}(z) &= F(\alpha', \beta', \gamma'; z), \\
\bar{G}_{zero}^{(2)}(z) &= z^{1-\gamma'} F(\alpha' + 1 - \gamma', \beta' + 1 - \gamma', 2 - \gamma'; z), \\
\alpha' &= \frac{1 - i\epsilon}{2} + \frac{iM + 1/2}{2}, \quad \beta' = \frac{1 - i\epsilon}{2} - \frac{iM + 1/2}{2};
\end{aligned} \tag{85}$$

Thus, we have constructed the following four regular solutions

$$F_{non-zero}^{(1)}, \quad F_{non-zero}^{(2)}, \quad G_{non-zero}^{(1)}, \quad G_{non-zero}^{(2)};$$

Due to the known identity for hypergeometric functions

$$F(A, B, C; z) = (1 - z)^{C-A-B} F(C - A, C - B, C; z)$$

we readily conclude that there exist only two different ones

$$F_{non-zero}^{(1)} = F_{non-zero}^{(2)}, \quad G_{non-zero}^{(1)} = G_{non-zero}^{(2)}. \tag{86}$$

The same is true for zero-solutions

$$F_{zero}^{(1)} = F_{zero}^{(2)}, \quad G_{zero}^{(1)} = G_{zero}^{(2)}. \tag{87}$$

Due to

$$\begin{aligned}
F_{non-zero}^{(1)} &= F_0^{non-zero} (1 - z)^{-i\epsilon/2} F(a, b, c, z), \\
G_{zero}^{(2)} &= G_0^{zero} z^{1/2} (1 - z)^{(1-i\epsilon)/2} F(\alpha' + 1 - \gamma', \beta' + 1 - \gamma', 2 - \gamma', z), \\
a + 1 &= \alpha' + 1 - \gamma', \quad b + 1 = \beta' + 1 - \gamma', \quad c + 1 = 2 - \gamma',
\end{aligned} \tag{88}$$

we can assume relationship

$$2\sqrt{z(1-z)} \left(\frac{d}{dz} - \frac{i\epsilon/2}{1-z} \right) F_{non-zero}^{(1)} + (M + \epsilon - i/2) G_{zero}^{(2)} = 0. \tag{89}$$

Indeed, from (89) we readily derive

$$\begin{aligned}
2\frac{ab}{c} F_0^{non-zero} + (M + \epsilon - i/2) G_0^{zero} &= 0 \implies \\
a F_0^{non-zero} + ic G_0^{zero} &= 0.
\end{aligned} \tag{90}$$

And similarly, due to

$$\begin{aligned}
G_{non-zero}^{(1)} &= G_0^{non-zero} (1 - z)^{+i\epsilon/2} F(a', b', c', z), \\
F_{zero}^{(2)} &= F_0^{zero} z^{1/2} (1 - z)^{(1+i\epsilon)/2} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, z), \\
a' + 1 &= \alpha + 1 - \gamma, \quad b' + 1 = \beta + 1 - \gamma, \quad c' + 1 = 2 - \gamma,
\end{aligned} \tag{91}$$

we can expect

$$2\sqrt{z(1-z)} \left(\frac{d}{dz} + \frac{i\epsilon/2}{1-z} \right) G_{non-zero}^{(1)} + (M - \epsilon - i/2) F_{zero}^{(2)} = 0 ; \quad (92)$$

from (92) it follows

$$\begin{aligned} 2\frac{a'b'}{c'} G_0^{non-zero} + (M - \epsilon - i/2) F_0^{zero} &= 0 \implies \\ a' G_0^{non-zero} + ic' F_0^{zero} &= 0 . \end{aligned} \quad (93)$$

6 Behavior of the solutions at the horizon, standing and running waves, $j > j_{\min}$

First, consider a couple of linearly independent solutions $F_{reg}^{(1)}$ and $F_{sing}^{(1)}$. It is convenient to introduce two other solutions with the help of the known Kummer's relation

$$\begin{aligned} U_1 &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} U_2 + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} U_6 , \\ U_5 &= \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} U_2 + \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} U_6 , \end{aligned} \quad (94)$$

and inverse ones

$$\begin{aligned} U_2 &= \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} U_1 + \frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} U_5 , \\ U_6 &= \frac{\Gamma(c+1-a-b)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} U_1 + \frac{\Gamma(c+1-a-b)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)} U_5 , \end{aligned} \quad (95)$$

where two couples of linearly independent solutions are involved:

$$\begin{aligned} U_1(z) &= F(a, b, c; z) , \\ U_5 &= z^{1-c} F(a+1-c, b+1-c, 2-c, z) ; \\ U_2(z) &= F(a, b, a+b-c+1; 1-z) , \\ U_6(z) &= (1-z)^{c-a-b} F(c-a, c-b, c-a-b+1; 1-z) . \end{aligned} \quad (96)$$

Applying relation (94) to the wave

$$F_{reg}^{(1)}(z) = F_{reg}^{(2)}(z) = F_{reg} = z^{(\nu+1)/2} (1-z)^{-i\epsilon/2} U_1 ,$$

we obtain

$$\begin{aligned} F_{reg}(z) &= z^{(\nu+1)/2} (1-z)^{-i\epsilon/2} U_1 = z^{(1+\nu)/2} (1-z)^{-i\epsilon/2} \\ &\times \left\{ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b, a+b-c+1; 1-z) \right. \\ &\left. + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{+i\epsilon+1/2} F(c-a, c-b, c-a-b+1; 1-z) \right\} , \end{aligned}$$

so that

$$F_{reg}(z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F_{out} + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F_{in} . \quad (97)$$

Here two independent solutions with simple behavior on the horizon are noted according to

$$\begin{aligned} F_{out} &= z^{(\nu+1)/2} (1-z)^{-i\epsilon/2} U_2 , & F_{in} &= z^{(\nu+1)/2} (1-z)^{-i\epsilon/2} U_6 \\ &= z^{(\nu+1)/2} (1-z)^{(+i\epsilon+1)/2} F(c-a, c-b, c-a-b+1; 1-z) . \end{aligned} \quad (98)$$

Doing the same for singular solutions

$$F_{sing}^{(1)} = F_{sing}^{(2)} = F_{sing} = z^{(\nu+1)/2} (1-z)^{-i\epsilon/2} U_5 ,$$

we get

$$\begin{aligned} F_{sing} &= z^{(\nu+1)/2} (1-z)^{-i\epsilon/2} U_5 = z^{(\nu+1)/2} (1-z)^{-i\epsilon/2} \\ &\times \left\{ \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} F(a, b, a+b-c+1; 1-z) \right. \\ &\quad + \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} (1-z)^{+i\epsilon+1/2} \\ &\quad \left. \times F(c-a, c-b, c-a-b+1; 1-z) \right\} , \end{aligned}$$

so that

$$F_{sing} = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} F_{out} + \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} F_{in} . \quad (99)$$

In a similar way let us consider solutions $G_{reg}^{(1)}(z) = G_{reg}^{(2)}(z) = G_{reg}$. Note that in relevant Kummer's formulas we use V instead of U . Thus we get

$$\begin{aligned} G_{reg}(z) &= z^{\nu/2} (1-z)^{+i\epsilon/2} V_1 = z^{\nu/2} (1-z)^{+i\epsilon/2} \\ &\times \left\{ \frac{\Gamma(c')\Gamma(c'-a'-b')}{\Gamma(c'-a')\Gamma(c'-b')} F(a', b', a'+b'-c'+1; 1-z) \right. \\ &\quad + \frac{\Gamma(c')\Gamma(a'+b'-c')}{\Gamma(a')\Gamma(b')} (1-z)^{-i\epsilon+1/2} \\ &\quad \left. \times F(c'-a', c'-b', c'-a'-b'+1; 1-z) \right\} \implies \end{aligned}$$

$$G_{reg}(z) = \frac{\Gamma(c')\Gamma(c'-a'-b')}{\Gamma(c'-a')\Gamma(c'-b')} G_{in} + \frac{\Gamma(c')\Gamma(a'+b'-c')}{\Gamma(a')\Gamma(b')} G_{out} , \quad (100)$$

where

$$\begin{aligned} G_{in} &= z^{\nu/2} (1-z)^{+i\epsilon/2} V_2 \\ &= z^{\nu/2} (1-z)^{+i\epsilon/2} z^{\nu/2} (1-z)^{+i\epsilon/2} F(a', b', a'+b'-c'+1; 1-z) , \\ G_{out} &= z^{\nu/2} (1-z)^{+i\epsilon/2} V_6 \\ &= z^{\nu/2} (1-z)^{(-i\epsilon+1)/2} F(c'-a', c'-b', c'-a'-b'+1; 1-z) . \end{aligned} \quad (101)$$

For singular ones, $G_{sing}^{(1)} = G_{sing}^{(2)} = G_{sing}$

$$G_{sing}^{(1)} = z^{\nu/2}(1-z)^{+i\epsilon/2}V_5 = z^{\nu/2}(1-z)^{+i\epsilon/2} \\ \times \left\{ \frac{\Gamma(2-c')\Gamma(c'-a'-b')}{\Gamma(1-a')\Gamma(1-b')} F(a', b', a'+b'-c'+1; 1-z) \right. \\ \left. + \frac{\Gamma(2-c')\Gamma(a'+b'-c')}{\Gamma(a'+1-c')\Gamma(b'+1-c')} \right. \\ \left. \times (1-z)^{-i\epsilon+1/2} F(c'-a', c'-b', c'-a'-b'+1; 1-z) \right\}$$

so that

$$G_{sing}^{(1)} = \frac{\Gamma(2-c')\Gamma(c'-a'-b')}{\Gamma(1-a')\Gamma(1-b')} G_{in} + \frac{\Gamma(2-c')\Gamma(a'+b'-c')}{\Gamma(a'+1-c')\Gamma(b'+1-c')} G_{out} . \quad (102)$$

It should be mentioned that the factors $(1-z)^{\pm i\epsilon/2}$ can be presented like plane waves. Indeed, let a new variable x be

$$1-z = e^{-2x}, \quad x = -\frac{1}{2} \ln(1-z), \quad x \in [0, +\infty) , \\ \text{then} \quad (1-z)^{-i\epsilon/2} = e^{i\epsilon x}, \quad (1-z)^{+i\epsilon/2} = e^{-i\epsilon x}, \quad x \rightarrow +\infty . \quad (103)$$

Evidently the *out*- and *in*-waves can be presented as linear combinations of *reg*- and *sing*-waves. Relevant relations are

$$F_{out} = z^{(\nu+1)/2}(1-z)^{-i\epsilon/2}U_2 = z^{(\nu+1)/2}(1-z)^{-i\epsilon/2} \\ \times \left(\frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} U_1 + \frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} U_5 \right) \\ = \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} F_{reg} + \frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} F_{sing} , \quad (104)$$

$$F_{in} = z^{(\nu+1)/2}(1-z)^{-i\epsilon/2}U_6 = z^{(\nu+1)/2}(1-z)^{-i\epsilon/2} \\ \times \left(\frac{\Gamma(c+1-a-b)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} U_1 + \frac{\Gamma(c+1-a-b)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)} U_5 \right) \\ = \frac{\Gamma(c+1-a-b)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} F_{reg} + \frac{\Gamma(c+1-a-b)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)} F_{sing} , \quad (105)$$

$$G_{in} = z^{\nu/2}(1-z)^{+i\epsilon/2}V_2 = z^{\nu/2}(1-z)^{+i\epsilon/2} \\ \times \left(\frac{\Gamma(a'+b'+1-c')\Gamma(1-c')}{\Gamma(a'+1-c')\Gamma(b'+1-c')} V_1 + \frac{\Gamma(a'+b'+1-c')\Gamma(c'-1)}{\Gamma(a')\Gamma(b')} V_5 \right) \\ = \frac{\Gamma(a'+b'+1-c')\Gamma(1-c')}{\Gamma(a'+1-c')\Gamma(b'+1-c')} G_{reg} + \frac{\Gamma(a'+b'+1-c')\Gamma(c'-1)}{\Gamma(a')\Gamma(b')} G_{sing} , \quad (106)$$

$$\begin{aligned}
G_{out} &= z^{\nu/2} (1-z)^{+i\epsilon/2} V_6 = z^{\nu/2} (1-z)^{+i\epsilon/2} \\
&\times \left(\frac{\Gamma(c' + 1 - a' - b') \Gamma(1 - c')}{\Gamma(1 - a') \Gamma(1 - b')} V_1 + \frac{\Gamma(c' + 1 - a' - b') \Gamma(c' - 1)}{\Gamma(c' - a') \Gamma(c' - b')} V_5 \right) \\
&= \frac{\Gamma(c' + 1 - a' - b') \Gamma(1 - c')}{\Gamma(1 - a') \Gamma(1 - b')} G_{reg} + \frac{\Gamma(c' + 1 - a' - b') \Gamma(c' - 1)}{\Gamma(c' - a') \Gamma(c' - b')} G_{sing} .
\end{aligned} \tag{107}$$

7 Standing and running waves at $j = j_{\min}$

Let write down results we need to proceed further

$$\begin{aligned}
F_{non-zero} &= (1-z)^{-i\epsilon/2} U_1 , & F_{zero} &= (1-z)^{-i\epsilon/2} U_5 , \\
G_{non-zero} &= (1-z)^{+i\epsilon/2} V_1 , & G_{zero} &= (1-z)^{+i\epsilon/2} V_5 ,
\end{aligned} \tag{108}$$

$$\begin{aligned}
U_1 &= F(a, b, c, z) , & V_1 &= F(a', b', c', z) , \\
a &= \frac{-i\epsilon}{2} + \frac{iM + 1/2}{2} , & b &= \frac{-i\epsilon}{2} - \frac{iM + 1/2}{2} , & c &= 1/2 , \\
a' &= \frac{+i\epsilon}{2} + \frac{iM + 1/2}{2} , & b' &= \frac{+i\epsilon}{2} - \frac{iM + 1/2}{2} , & c' &= 1/2 .
\end{aligned} \tag{109}$$

We readily derive

$$\begin{aligned}
F_{non-zero} &= (1-z)^{-i\epsilon/2} U_1 \\
&= \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} F_{out} + \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} F_{in} , \\
F_{zero} &= (1-z)^{-i\epsilon/2} U_5 \\
&= \frac{\Gamma(2 - c) \Gamma(c - a - b)}{\Gamma(1 - a) \Gamma(1 - b)} F_{out} + \frac{\Gamma(2 - c) \Gamma(a + b - c)}{\Gamma(a + 1 - c) \Gamma(b + 1 - c)} F_{in} , \\
F_{out} &= (1-z)^{-i\epsilon/2} U_2 , & F_{in} &= (1-z)^{-i\epsilon/2} U_6 .
\end{aligned} \tag{110}$$

$$\begin{aligned}
G_{non-zero} &= (1-z)^{+i\epsilon/2} V_1 \\
&= \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} G_{in} + \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} G_{out} , \\
G_{zero} &= (1-z)^{+i\epsilon/2} V_5 \\
&= \frac{\Gamma(2 - c') \Gamma(c' - a' - b')}{\Gamma(1 - a') \Gamma(1 - b')} G_{in} + \frac{\Gamma(2 - c') \Gamma(a' + b' - c')}{\Gamma(a' + 1 - c') \Gamma(b' + 1 - c')} G_{out} , \\
G_{in} &= (1-z)^{+i\epsilon/2} V_2 , & G_{out} &= (1-z)^{+i\epsilon/2} V_6 .
\end{aligned} \tag{111}$$

8 Discussion and conclusions

To understand better the situation, let us consider the case of minimal j_{\min} in the limit of vanishing curvature. It is convenient to start with the first order systems for minimal values j_{\min} in the case of Minkowski space:

$$k = +1/2, +1, \dots$$

$$\begin{aligned} \epsilon f_3 - i \frac{d}{dr} f_3 - M f_1 &= 0 , \\ \epsilon f_1 + i \frac{d}{dr} f_1 - M f_3 &= 0 ; \end{aligned} \quad (112)$$

$$k = -1/2, -1, \dots$$

$$\begin{aligned} \epsilon f_4 + i \frac{d}{dr} f_4 - M f_2 &= 0 , \\ \epsilon f_2 - i \frac{d}{dr} f_2 - M f_4 &= 0 . \end{aligned} \quad (113)$$

Let us detail the case of positive $k = +1/2, +1, \dots$. With notation

$$\frac{f_1 + f_3}{\sqrt{2}} = h(r) , \quad \frac{f_1 - f_3}{i\sqrt{2}} = g(r) ; \quad (114)$$

relevant equations are

$$\frac{d}{dr} h + (\epsilon + M) g = 0 , \quad \frac{d}{dr} g - (\epsilon - M) h = 0 . \quad (115)$$

Further, with the substitutions

$$h(r) = H e^{\gamma r} , \quad g(r) = G e^{\gamma r} \quad (116)$$

we get (first let it be $(\epsilon^2 - M^2) > 0$)

$$\begin{aligned} \gamma^2 &= -(\epsilon^2 - M^2) = -p^2 , \\ \gamma &= +ip, -ip , \quad G\gamma - (\epsilon - M)H = 0 \end{aligned} \quad (117)$$

Thus we have two linearly independent solutions

$$h_1(r) = H_1 e^{+ipr} , \quad g_1(r) = G_1 e^{+ipr} , \quad G_1 = \frac{\epsilon - M}{ip} H_1 ; \quad (118)$$

and

$$h_2(r) = H_2 e^{-ipr} , \quad g_2(r) = G_2 e^{-ipr} , \quad G_2 = \frac{\epsilon - M}{-ip} H_2 ; \quad (119)$$

for simplicity, we will take $H_1 = H_2 = 1$. It is convenient to introduce linear combinations of these solutions

the first

$$\begin{aligned}\frac{h_1(r) + h_2(r)}{2} &= \cos pr , \\ \frac{g_1(r) + g_2(r)}{2} &= \frac{\epsilon - M}{p} \sin pr ;\end{aligned}\tag{120}$$

the second

$$\begin{aligned}\frac{h_1(r) - h_2(r)}{2i} &= \sin pr , \\ \frac{g_1(r) - g_2(r)}{2i} &= \frac{\epsilon - M}{-p} \cos pr .\end{aligned}\tag{121}$$

Now let us specify the case $(\epsilon^2 - M^2) < 0$:

$$\begin{aligned}\gamma^2 &= -(\epsilon^2 - M^2) \equiv +q^2 , \quad \gamma = +q, -q , \\ G\gamma - (\epsilon - M)H &= 0 .\end{aligned}\tag{122}$$

We have two linearly independent solutions

$$h_1(r) = H_1 e^{+qr} , \quad g_1(r) = G_1 e^{+qr} , \quad G_1 = \frac{\epsilon - M}{q} H_1 ;\tag{123}$$

$$h_2(r) = H_2 e^{-qr} , \quad g_2(r) = G_2 e^{-qr} , \quad G_2 = \frac{\epsilon - M}{-q} H_2 .\tag{124}$$

Below, $H_1 = H_2 = 1$. We can introduce two linear combinations of these solutions

the first

$$\begin{aligned}\frac{h_1(r) + h_2(r)}{2} &= \cosh qr , \\ \frac{g_1(r) + g_2(r)}{2} &= \frac{\epsilon - M}{q} \sinh qr\end{aligned}\tag{125}$$

the second

$$\begin{aligned}\frac{h_1(r) - h_2(r)}{2} &= \sinh qr , \\ \frac{g_1(r) - g_2(r)}{2} &= \frac{\epsilon - M}{q} \cosh qr .\end{aligned}\tag{126}$$

Evidently, above constructed solutions in de Sitter model provide us with generalizations of these in Minkowski space. It may be verified additionally by direct limiting process when $\rho \rightarrow \infty$. To this end, let us translate solutions in de Sitter space to usual units (ρ is the curvature radius, E is the energy, c is the light velocity, m is the electron mass)

$$\begin{aligned}F_{non-zero}(R) &= \left(1 - \frac{R^2}{\rho^2}\right)^{-i\frac{E\rho}{2c\hbar}} F(a, b, c; \frac{R^2}{\rho^2}) , \\ F_{zero}(R) &= R \left(1 - \frac{R^2}{\rho^2}\right)^{+i\frac{E\rho}{2c\hbar} + 1/2} F(a + 1 - c, b + 1 - c, 2 - c; \frac{R^2}{\rho^2}) ,\end{aligned}$$

$$G_{non-zero}(R) = \left(1 - \frac{R^2}{\rho^2}\right)^{+i\frac{E\rho}{2c\hbar}} F(a', b', c; \frac{R^2}{\rho^2}) ,$$

$$G_{zero}(R) = R \left(1 - \frac{R^2}{\rho^2}\right)^{-i\frac{E\rho}{2c\hbar}+1/2} F(a' + 1 - c, b' + 1 - c, 2 - c; \frac{R^2}{\rho^2}) ,$$

Parameters of hypergeometric functions are given by

$$c = \frac{1}{2} , \quad a = \frac{1}{2} \left[+1/2 + i\left(\frac{mc\rho}{\hbar} - \frac{E\rho}{c\hbar}\right) \right] , \quad b = \frac{1}{2} \left[-i\left(\frac{mc\rho}{\hbar} + \frac{E\rho}{c\hbar}\right) - 1/2 \right] ,$$

$$c = \frac{1}{2} , \quad a' = \frac{1}{2} \left[+1/2 + i\left(\frac{mc\rho}{\hbar} + \frac{E\rho}{c\hbar}\right) \right] , \quad b' = \frac{1}{2} \left[-i\left(\frac{mc\rho}{\hbar} - \frac{E\rho}{c\hbar}\right) - 1/2 \right] .$$

Let us examine the limiting procedure at $\rho \rightarrow \infty$ in $F(a, b, c; R^2/\rho^2)$. Because

$$\begin{aligned} \frac{1}{1!} \frac{ab}{c} \frac{R^2}{\rho^2} &\rightarrow \frac{1}{2!} (m^2 c^2 / \hbar^2 - E^2 / \hbar^2 c^2) R^2 = -\frac{1}{2!} (pR)^2 , \\ \frac{1}{2!} \frac{a(a+1)b(b+1)}{c(c+1)} \frac{R^2}{\rho^2} &\rightarrow +\frac{(pR)^4}{4!} , \\ \frac{1}{3!} \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)} \frac{R^2}{\rho^2} &\rightarrow -\frac{(pR)^6}{6!} , \end{aligned}$$

and so on, we obtain the following limiting relation

$$\begin{aligned} \lim_{\rho \rightarrow \infty} F(a, b, c; \frac{R^2}{\rho^2}) &= \cos pr \quad \implies \\ \lim_{\rho \rightarrow \infty} F_{non-zero}(R) &= \cos pr , \quad \lim_{\rho \rightarrow \infty} G_{non-zero}(R) = \cos pr . \end{aligned} \tag{127}$$

In the same manner, let us examine the function

$$\begin{aligned} R F(a + 1 - c, b + 1 - c, 2 - c; \frac{R^2}{\rho^2}) , \\ A = a + 1 - c = \frac{3/2 + i(M + \epsilon)}{2} , \\ B = b + 1 - c = \frac{1/2 - i(M - \epsilon)}{2} , \quad C = 3/2 . \end{aligned}$$

Taking into account

$$\begin{aligned} \frac{AB}{C} &\implies -\frac{1}{3!} (pR)^2 , \quad \frac{1}{2!} \frac{A(A+1)B(B+1)}{C(C+1)} \implies +\frac{1}{5!} (pR)^4 , \\ \frac{1}{3!} \frac{A(A+1)(A+2)B(B+1)(B+2)}{C(C+1)(C+2)} &\implies -\frac{1}{7!} (pR)^6 , \end{aligned}$$

and so on, we arrive at the relationships

$$\begin{aligned} \lim_{\rho \rightarrow \infty} pR \ F(a+1-c, b+1-c, 2-c; \frac{R^2}{\rho^2}) = \sin pR & \implies \\ \lim_{\rho \rightarrow \infty} pR \ F_{zero} = \sin pR, & \quad \lim_{\rho \rightarrow \infty} pR \ G_{zero} = \sin pR. \end{aligned} \quad (128)$$

Thus, indeed, solutions in de Sitter model are extensions of more simple and well-known ones in Minkowski model.

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